University of California, Berkeley
Physics 105 Fall 2000 Section 2 (*Strovink*)

SOLUTION TO PROBLEM SET 4
Solutions by J. Barber and T. Bunn

Reading:
105 Notes 6.1-6.2, 3.1-3.3
Hand & Finch 3.1-3.3

1.
Generalize the Euler equation

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} = \frac{\partial \mathcal{L}}{\partial y}$$

to the case in which $\mathcal{L}$ is a function of t, y, $\dot{y}$, *and* $\ddot{y}$. Derive the new Euler equation for this case. Assume that $y(t_1)$, $y(t_2)$, *and* $\dot{y}(t_1)$, $\dot{y}(t_2)$ are not varied, *i.e.* both the value and the slope of y are fixed at each endpoint.
[*Hint*: Compared to the derivation of the usual Euler equation, when you calculate the variation of the action J with the parameter α, you will have an extra term in the integrand. Integrate that term by parts *twice*.]
Solution:

$$J = \int_{t_1}^{t_2} \mathcal{L}(y, \dot{y}, \ddot{y}, t)dt$$

As usual, suppose $y = y(t, \alpha)$, where α is a parameter, and suppose $y(t, \alpha = 0)$ minimizes J. Thus:

$$\frac{dJ}{d\alpha} = \int_{t_1}^{t_2} \left(\frac{\partial \mathcal{L}}{\partial y}\frac{dy}{d\alpha} + \frac{\partial \mathcal{L}}{\partial \dot{y}}\frac{d\dot{y}}{d\alpha} + \frac{\partial \mathcal{L}}{\partial \ddot{y}}\frac{d\ddot{y}}{d\alpha} \right) dt = 0$$

Integrate the second term once by parts:

$$\begin{aligned}
&\int_{t_1}^{t_2} \frac{\partial \mathcal{L}}{\partial \dot{y}}\frac{d}{dt}\left(\frac{dy}{d\alpha}\right) dt \\
&= \frac{\partial \mathcal{L}}{\partial \dot{y}}\frac{dy}{d\alpha}\bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{dy}{d\alpha}\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) dt \\
&= 0 - \int_{t_1}^{t_2} \frac{dy}{d\alpha}\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) dt\ ,
\end{aligned}$$

where the '0' follows from the endpoints being fixed. Now integrate the second term by parts twice:

$$\begin{aligned}
&\int_{t_1}^{t_2} \frac{\partial \mathcal{L}}{\partial \ddot{y}}\frac{d\ddot{y}}{d\alpha} dt \\
&= \frac{\partial \mathcal{L}}{\partial \ddot{y}}\frac{d\dot{y}}{d\alpha}\bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \ddot{y}}\right)\frac{d}{dt}\left(\frac{dy}{d\alpha}\right) dt \\
&= \frac{\partial \mathcal{L}}{\partial \ddot{y}}\frac{d\dot{y}}{d\alpha}\bigg|_{t_1}^{t_2} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \ddot{y}}\right)\frac{dy}{d\alpha}\bigg|_{t_1}^{t_2} \\
&\quad + \int_{t_1}^{t_2} \frac{dy}{d\alpha}\frac{d^2}{dt^2}\left(\frac{\partial \mathcal{L}}{\partial \ddot{y}}\right) dt \\
&= 0 - 0 + \int_{t_1}^{t_2} \frac{dy}{d\alpha}\frac{d^2}{dt^2}\left(\frac{\partial \mathcal{L}}{\partial \ddot{y}}\right) dt
\end{aligned}$$

What remains is:

$$\begin{aligned}
\frac{dJ}{d\alpha} &= \int_{t_1}^{t_2} \left(\frac{\partial \mathcal{L}}{\partial y} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} + \frac{d^2}{dt^2}\frac{\partial \mathcal{L}}{\partial \ddot{y}} \right)\frac{dy}{d\alpha} dt \\
&= 0
\end{aligned}$$

The only way the above can be true for arbitrary variation with α is for the quantity in parenthesis to be zero. In other words:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) = \frac{\partial \mathcal{L}}{\partial y} + \frac{d^2}{dt^2}\left(\frac{\partial \mathcal{L}}{\partial \ddot{y}}\right)$$

It is easy to see what the pattern would be for a lagrangian depending on yet higher derivatives of y.

2.
A bead moves in a constant gravitational field $\mathbf{a} = \hat{\mathbf{x}}g$ with an initial velocity $|\mathbf{v}| = v_0$, where g and v_0 are positive constants. It is constrained to slide along a frictionless wire which has an unknown shape $y(x)$. (Notice that $\hat{\mathbf{x}}$ points down and $\hat{\mathbf{y}}$ points to the right in this problem.)

(**a**)
Show that the shape $y(x)$ which minimizes the bead's transit time between two fixed points $(0, 0)$ and (X, Y) is given by a set of parametric equations

$$x = x_0 + a(1 - \cos\phi)$$
$$y = y_0 + b(\phi - \sin\phi) \ ,$$

where ϕ is the parameter. This is the famous *brachistochrone problem*. The solution is a *cycloid* – the path of a dot painted on a rolling wheel.
Solution:
The transit time, starting from (X,Y) and ending up at (0,0), is given by:

$$T = \int_X^0 dt = \int_X^0 \frac{dl}{v} = \int_X^0 \frac{\sqrt{\left(\frac{dy}{dx}\right)^2 + 1}}{\sqrt{2g}\sqrt{\frac{E}{mg} + x}}\, dx$$

where the denominator in the last step follows from conservation of energy, with $E = \frac{1}{2}m{v_o}^2 - mgX$. Calling the integrand $\mathcal{L}$, we apply the Euler equation:

$$\frac{d}{dx}\left(\frac{\partial\mathcal{L}}{\partial y'}\right) = \frac{\partial\mathcal{L}}{\partial y} = 0$$

$$\frac{d}{dx}\left(\frac{y'}{\sqrt{y'^2+1}\sqrt{\frac{E}{mg}+x}}\right) = 0$$

$$\frac{y'}{\sqrt{y'^2+1}\sqrt{\frac{E}{mg}+x}} = C \ \ \text{(a constant)}$$

To show that the above parametric equations for $x(\phi)$ and $y(\phi)$ are the solution to this D.E., we'll need the following:

$$y' = \frac{dy}{dx} = \frac{dy}{d\phi}\frac{d\phi}{dx} = \frac{b}{a}\frac{(1-\cos\phi)}{\sin\phi}$$

Substituting this into the D.E. yields:

$$\frac{b}{a}\frac{(1-\cos\phi)}{\sin\phi\sqrt{\frac{b^2}{a^2}\frac{(1-\cos\phi)^2}{\sin^2\phi}+1}\sqrt{\frac{E}{mg}+x_o+a(1-\cos\phi)}} = C$$

$$\frac{(1-\cos\phi)}{\sqrt{1-2\cos\phi+\cos^2\phi+\frac{a^2}{b^2}\sin^2\phi}\sqrt{\frac{E}{mg}+x_o+a(1-\cos\phi)}} = C$$

If $a = b$ and $x_o = -\frac{E}{mg}$, then the above is true for all ϕ, as it reduces to:

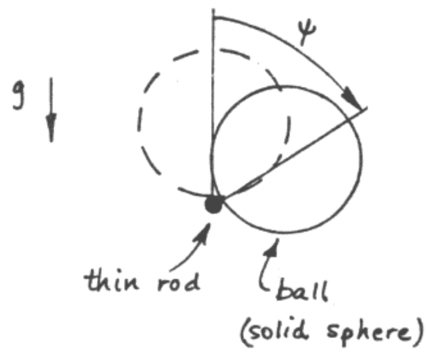
$$\frac{(1-\cos\phi)}{\sqrt{2(1-\cos\phi)}\sqrt{1-\cos\phi}\sqrt{a}} = \frac{1}{\sqrt{2a}} = C$$

which is a true statement.
(**b**)
In terms of v_0, g, X, and Y, what are the values of the constants x_0, y_0, a, and b which yield the optimal trajectory? Give definite answers where you can; otherwise provide equations which, if solved, would yield those values. [*Hint*: see Hand & Finch, problems 2.9 and 2.10.]
Solution:
Let's say that, at ϕ_i, (x,y)=(X,Y), and at ϕ_f (x,y)=(0,0). This, along with the conditions that a=b and $x_o = \frac{E}{mg}$ from part (a), yields the following equations:

$$X = -\frac{{v_o}^2}{2g} + X + a(1-\cos\phi_i)$$
$$Y = y_o + a(\phi_i - \sin\phi_i)$$
$$0 = -\frac{{v_o}^2}{2g} + X + a(1-\cos\phi_f)$$
$$0 = y_o + a(\phi_f - \sin\phi_f)$$

This is a set of four transcendental equations for four unknowns: a, y_o, ϕ_i, and ϕ_f.

3.
Starting from a vertical position at rest, a solid ball resting on top of a thin rod falls off.



While in contact with the rod, the ball rolls without slipping. Using the method of Lagrange undetermined multipliers, find the angle ψ at which the ball leaves the rod ($\psi \equiv 0$ initially).

Solution:
Let's pick as our coordinates ψ, the angle down from the y-axis that the ball has rolled (y points up), and r, the distance of the center of the sphere from the thin rod. (So we're working with the ordinary polar coordinates of the center of the sphere.) The equation of constraint is $r = R$, where R is the radius of the sphere. Then the kinetic energy is $T = \frac{1}{5}mR^2\dot{\psi}^2 + \frac{1}{2}mr^2\dot{\psi}^2 + \frac{1}{2}m\dot{r}^2$. (The first term represents the rotation of the sphere about its center, and the other two represent translations of the center of the sphere in the tangential and radial directions.) The potential energy is $V = mgr\cos\psi$. So

$$\mathcal{L} = \frac{1}{2}m\left(\frac{2}{5}R^2 + r^2\right)\dot{\psi}^2 + \frac{1}{2}m\dot{r}^2 - mgr\cos\psi$$

The Euler-Lagrange equation for r is

$$\ddot{r} - r\dot{\psi}^2 + g\cos\psi = \lambda$$

Use the equation of constraint to remove the $\ddot{r}$ from the above and replace r by the constant R: $\lambda = -R\dot{\psi}^2 + g\cos\psi$. We want to find the angle ψ_ℓ at which the force of constraint λ equals zero: $\dot{\psi}_\ell^2 = (g/R)\cos\psi_\ell$. Let's get rid of the $\dot{\psi}_\ell$. Energy conservation says

$$\frac{7}{10}mR^2\dot{\psi}^2 + mgR\cos\psi = mgR$$

Solve for $\dot{\psi}$: $\dot{\psi}^2 = \frac{10g}{7R}(1-\cos\psi)$. Substitute into the equation for ψ_ℓ, and you get

$$\cos\psi_\ell = \frac{10}{17}, \qquad \text{so} \qquad \psi_\ell \approx 54^\circ.$$

4.
Consider a simple, plane pendulum consisting of a mass m attached to a string of length l. Only small oscillations need be considered. After the pendulum is set into motion, the length of the string is shortened at a constant rate $dl/dt = -\alpha$, where $\alpha > 0$. (The string is pulled through a small hole located at a constant position, so the pendulum's suspension point remains fixed.) Compute the Lagrangian and Hamiltonian functions. Compare the Hamiltonian and the total energy of the pendulum, and discuss the conservation of energy for the system.

Solution:
The lagrangian is $\mathcal{L} = \frac{m}{2}\dot{\ell}^2 + \frac{m}{2}\ell^2\dot{\theta}^2 + mg\ell\cos\theta$. Use $\ell = \ell_0 - \alpha t$ to write this in terms of the variables $(\theta, \dot{\theta}, t)$:

$$\mathcal{L}(\theta,\dot{\theta},t) =$$
$$\frac{1}{2}m\alpha^2 + \frac{1}{2}m(\ell_0-\alpha t)^2\dot{\theta}^2 + mg(\ell_0-\alpha t)\cos\theta$$

The momentum canonically conjugate to the coordinate θ is $p = \frac{d\mathcal{L}}{d\dot{\theta}} = m(\ell_0-\alpha t)^2\dot{\theta}$, so we can write

$$\mathcal{H}(\theta,p,t) = p\dot{\theta} - \mathcal{L}$$

$$= \frac{1}{2}m(\ell_0-\alpha t)^2\dot{\theta}^2 - \frac{1}{2}m\alpha^2 - mg(\ell_0-\alpha t)\cos\theta$$
$$= \frac{p^2}{2m(\ell_0-\alpha t)^2} - \frac{1}{2}m\alpha^2 - mg(\ell_0-\alpha t)\cos\theta$$

This is not the same as the total energy. (The energy would have a + sign in the second term.) The rules about hamiltonians say that $\mathcal{H} = E$ if the generalized coordinates are related to ordinary cartesian coordinates in a way that doesn't depend explicitly on t, and if there are no velocity-dependent potentials. That first condition isn't satisfied here, so it's OK that $\mathcal{H} \neq E$. Energy is also not conserved in this system, since whoever is pulling up on the string is doing work on the system.

5.

A particle of mass m and velocity $\mathbf{v}_1$ leaves a semi-infinite space $z < 0$, where the potential energy is a constant U_1, and enters the remaining space $z > 0$, where the potential is a constant U_2.

(**a**)

Use symmetry arguments to find two constants of the motion.

Solution:

The lagrangian for this problem is

$$\mathcal{L} = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - U(z) \ ,$$

where U(z) is the potential energy, which clearly depends only on z. Applying the Euler-Lagrange equation:

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) &= \frac{\partial \mathcal{L}}{\partial x} = 0 \\
m\dot{x} &= \text{constant} \\
\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) &= \frac{\partial \mathcal{L}}{\partial y} = 0 \\
m\dot{y} &= \text{constant}
\end{aligned}$$

(**b**)

Use these two constants to obtain the new velocity $\mathbf{v}_2$.

Solution:

By the constants of part (a), $v_{2x} = v_{1x}$ and $v_{2y} = v_{1y}$. Since the lagrangian does not depend explicitly on time, we know that the total energy is conserved as well:

$$\begin{aligned}
&\tfrac{1}{2}m{v_1}^2 + U_1 = \tfrac{1}{2}m{v_2}^2 + U_2 \\
&\frac{m}{2}\left({v_{1x}}^2 + {v_{1y}}^2 + {v_{1z}}^2\right) + U_1 \\
&\qquad = \frac{m}{2}\left({v_{2x}}^2 + {v_{2y}}^2 + {v_{2z}}^2\right) + U_2 \\
&\frac{m}{2}{v_{1z}}^2 + U_1 = \frac{m}{2}{v_{2z}}^2 + U_2 \\
&\qquad v_{2z} = \sqrt{{v_{1z}}^2 + \frac{2}{m}\left(U_1 - U_2\right)}
\end{aligned}$$

We take the positive root, for if v_{2z} were negative, then the particle would not travel into the positive z space at all. Hence:

$$\mathbf{v}_2 = v_{1x}\ \hat{i} + v_{1y}\ \hat{j} + \sqrt{{v_{1z}}^2 + \frac{2}{m}\left(U_1 - U_2\right)}\ \hat{k}$$

6.

The Lagrangian for a (physically interesting) system is

$$\begin{aligned}
&\mathcal{L}(\varphi, \dot{\varphi}, \theta, \dot{\theta}, \psi, \dot{\psi}, t) = \tfrac{1}{2}I(\dot{\varphi}^2 \sin^2\theta + \dot{\theta}^2) \\
&+ \tfrac{1}{2}I_3(\dot{\varphi}\cos\theta + \dot{\psi})^2 - mgh\cos\theta \ ,
\end{aligned}$$

where (φ, θ, ψ) are Euler angles and (I, I_3, mgh) are constants.

(**a**)

Find two cyclic coordinates and obtain the two corresponding conserved canonically conjugate momenta.

Solution:

$\mathcal{L}$ is independent of φ and ψ, so they are cyclic.

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \dot{\varphi}} &= I\dot{\varphi}\sin^2\theta + I_3\left(\dot{\varphi}\cos\theta + \dot{\psi}\right)\cos\theta \\
&= p_\varphi \quad \text{(a constant)} \\
\frac{\partial \mathcal{L}}{\partial \dot{\psi}} &= I_3\left(\dot{\varphi}\cos\theta + \dot{\psi}\right) \\
&= p_\psi \quad \text{(a constant)}
\end{aligned}$$

(**b**)

Find a third constant of the motion.

Solution:

Since $\mathcal{L}$ does not depend explicitly on time, the Hamiltonian is equal to the total energy, and is thus conserved:

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}}\dot{\varphi} + \frac{\partial \mathcal{L}}{\partial \dot{\psi}}\dot{\psi} + \frac{\partial \mathcal{L}}{\partial \dot{\theta}}\dot{\theta} - \mathcal{L}$$

Inserting $\mathcal{L}$ and the other expressions from above, and simplifying, yields:

$$\begin{aligned}
\mathcal{H} = &\tfrac{1}{2}I\left(\dot{\varphi}^2\sin^2\theta + \dot{\theta}^2\right) + \\
&\tfrac{1}{2}I_3\left(\dot{\varphi}\cos\theta + \dot{\psi}\right)^2 + mgh\cos\theta \\
= &\ \text{constant}
\end{aligned}$$

(**c**)
Using the results of (**a**) and (**b**), express $\dot{\theta}^2$ as a function only of θ and constants.
Solution:
From our expressions in (a) we find that

$$\dot{\varphi} = \frac{p_\varphi - p_\psi \cos\theta}{I \sin^2\theta}$$
$$\dot{\psi} = \frac{p_\psi}{I_3} - \frac{(p_\varphi - p_\psi \cos\theta)\cos\theta}{I \sin^2\theta}$$

Substituting this into our expression for $\mathcal{H}$ yields:

$$\mathcal{H} = \tfrac{1}{2} I \left(\frac{(p_\varphi - p_\psi \cos\theta)^2}{I^2 \sin^2\theta} + \dot{\theta}^2 \right) + \tfrac{1}{2} I_3 \left(\frac{p_\psi}{I_3} \right)^2 + mgh\cos\theta$$

Solve for $\dot{\theta}^2$:

$$\dot{\theta}^2 = \frac{2\mathcal{H}}{I} - \frac{p_\psi^2}{I I_3} - \frac{2mgh\cos\theta}{I} - \frac{(p_\varphi - p_\psi\cos\theta)^2}{I^2 \sin^2\theta}$$

7.
The interaction Lagrangian for a system consisting of a relativistic test particle of mass m and charge e moving in a static electromagnetic field is

$$\mathcal{L}(\mathbf{x}, \mathbf{v}, t) = -mc^2 \sqrt{1 - \frac{v^2}{c^2}} + e\mathbf{v} \cdot \mathbf{A} - e\phi \ ,$$

where $\mathbf{x}$ is the particle's position, $\mathbf{v}$ is its velocity, $\phi(\mathbf{x})$ is the electrostatic potential ($\mathbf{E} = -\nabla\phi$), $\mathbf{A}$ is the (static) magnetic vector potential ($\mathbf{B} = \nabla \times \mathbf{A}$), and c is the speed of light.

(**a**)
Write down the canonical momenta (p_1, p_2, p_3) which are conjugate to the Cartesian coordinates (x_1, x_2, x_3).
Solution:

$$p_i = \frac{\partial \mathcal{L}}{\partial v_i} = \frac{m v_i}{\sqrt{1 - \frac{v^2}{c^2}}} + eA_i$$

(**b**)
Compute the Hamiltonian $\mathcal{H}(\mathbf{x}, \mathbf{v}, t)$.
Solution:

$$\begin{aligned}\mathcal{H} &= p_i v_i - \mathcal{L} \\ &= \frac{mv^2}{\sqrt{1 - \frac{v^2}{c^2}}} + e\mathbf{v}\cdot\mathbf{A} \\ &\quad - \left(-mc^2 \sqrt{1 - \frac{v^2}{c^2}} + e\mathbf{v}\cdot\mathbf{A} - e\phi \right) \\ &= \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} + e\phi\end{aligned}$$

(**c**)
Re-express $\mathcal{H}(\mathbf{x}, \mathbf{v}, t)$ as the function $\mathcal{H}(\mathbf{x}, \mathbf{p}, t)$.
Solution:
From part (a), we can solve for the v's in terms of the p's:

$$v_i = \frac{c(p_i - eA_i)}{\sqrt{m^2c^2 + (\mathbf{p} - e\mathbf{A})^2}}$$

Inserting this into our answer to (b):

$$\begin{aligned}\mathcal{H} &= \frac{mc^2}{\sqrt{1 - \frac{1}{2}\left(c^2 \frac{(\mathbf{p} - e\mathbf{A})^2}{m^2c^2 + (\mathbf{p} - e\mathbf{A})^2} \right)}} + e\phi \\ &= \sqrt{m^2c^4 + (\mathbf{p} - e\mathbf{A})^2 c^2} + e\phi\end{aligned}$$

(**d**)
Show that $\mathcal{H}$ is conserved. Is it equal to

$$E = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} \ ,$$

the total (relativistic) energy of the test particle? Explain.
Solution:
Since $\mathcal{L}$ does not depend explicitly on t, $\frac{d\mathcal{H}}{dt} = 0$, and so $\mathcal{H}$ is conserved. However, $\mathcal{H}$ is *not* equal to the above expression for relativistic energy. This is because the above expression is derived assuming no external fields, such as the electric field generated by the ϕ which appears in our $\mathcal{H}$.

8.
Consider f and g to be any two continuous functions of the generalized coordinates q_i and canonically conjugate momenta p_i, as well as time:

$$f = f(q_i, p_i, t)$$
$$g = g(q_i, p_i, t)\ .$$

The *Poisson bracket* of f and g is defined by

$$[f, g] \equiv \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\ ,$$

where summation over i is implied. Prove the following properties of the Poisson bracket:
(**a**)

$$\frac{df}{dt} = [f, \mathcal{H}] + \frac{\partial f}{\partial t}$$

Solution:

$$[f, H] = \frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i} = \frac{\partial f}{\partial q_i}\frac{dq_i}{dt} + \frac{\partial f}{\partial p_i}\frac{dp_i}{dt}$$

where the last step comes from Hamilton's equations of motion. By the chain rule,

$$\frac{df}{dt} = \frac{\partial f}{\partial q_i}\frac{dq_i}{dt} + \frac{\partial f}{\partial p_i}\frac{dp_i}{dt} + \frac{\partial f}{\partial t}$$

Compare these last two equations, and you see that $df/dt = [f, H] + \partial f/\partial t$.

(**b**)

$$\dot{q}_i = [q_i, \mathcal{H}]$$

Solution:

$$[q_i, H] = \frac{\partial q_i}{\partial q_k}\frac{\partial H}{\partial p_k} - \frac{\partial q_i}{\partial p_k}\frac{\partial H}{\partial q_k}$$

But $\partial q_i/\partial p_k = 0$ (p's and q's are independent variables), and $\partial q_i/\partial q_k = \delta_{ik}$. So only one term in the sum over k is nonzero:

$$[q_i, H] = \delta_{ik}\frac{\partial H}{\partial p_k} = \frac{\partial H}{\partial p_i} = \dot{q}_i$$

(by Hamilton's equations again.)

(**c**)

$$\dot{p}_i = [p_i, \mathcal{H}]$$

Solution:
Just like part (b): $[p_i, H] = \frac{\partial p_i}{\partial q_k}\frac{\partial H}{\partial p_k} - \frac{\partial p_i}{\partial p_k}\frac{\partial H}{\partial q_k} = -\delta_{ik}\frac{\partial H}{\partial q_k} = -\frac{\partial H}{\partial q_i} = \dot{p}_i$.

(**d**)

$$[p_i, p_j] = 0$$

Solution:
$[p_i, p_j] = \frac{\partial p_i}{\partial q_k}\frac{\partial p_j}{\partial p_k} - \frac{\partial p_i}{\partial p_k}\frac{\partial p_j}{\partial q_k}$. But each term in this expression contains a $\partial p/\partial q$, and that's zero. So the whole expression is zero.

(**e**)

$$[q_i, q_j] = 0$$

Solution:
Same as part (d): Each term in the Poisson bracket contains a $\partial q/\partial p$, which is zero.

(**f**)

$$[q_i, p_j] = \delta_{ij}\ ,$$

where $\mathcal{H}$ is the Hamiltonian. If the Poisson bracket of two quantities is equal to unity, the quantities are said to be *canonically conjugate*. On the other hand, if the Poisson bracket vanishes, the quantities are said to *commute*.
Solution:

$$[q_i, p_j] = \frac{\partial q_i}{\partial q_k}\frac{\partial p_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k}\frac{\partial p_j}{\partial q_k} = \delta_{ik}\delta_{jk} - 0\cdot 0 = \delta_{ij}$$

(**g**)
Show that any quantity that does not depend explicitly on the time and that commutes with the Hamiltonian is a constant of the motion.
Solution:
"Constant of the motion" means $df/dt = 0$, so what we need to show is that if $[f, H] = \partial f/\partial t = 0$, then $df/dt = 0$. But that follows immediately from part (a), by simply substituting for $[f, H]$ and $\partial f/\partial t$.